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Periodic Optimal Flight

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For a host of dynamic systems, periodic motion is more efficient than steady-state operation. This work focuses on atmospheric fuel-efficient periodic flight. To understand some of the generic properties of periodic optimal flight, the theoretical aspects are presented, which form the basis for the numerical computation of periodic optimal paths. Furthermore, the simplest problem for which periodic optimal control can be induced is formulated and various solutions based on asymptotic analysis are given. Intuitive physical mechanisms that contribute to producing periodic optimal flight are discussed on the basis of various levels of approximation of the aircraft dynamic models and upon perturbation analysis about the steady-state cruise path. The performance improvement and the dramatic periodic optimal flight path of a hypersonic vehicle are computed. To mechanize this flight path, a neighboring optimum perturbation guidance law is developed and its performance is presented.

I. Introduction

IN nature, numerous examples of cyclic processes are found, ranging from the orbital motion of the heavenly bodies to the rhythm of the heart. In contrast, many engineering systems are designed to operate in a steady-state mode, although performance might be vastly improved by cyclic operation. The idea of periodic control can be traced back at least to Hausen in 1927, who considered the cyclic operation of heat regenerators. In the 1940s, Sänger suggested that cruise fuel efficiency could be improved by periodic motion of an aircraft and Edelbaum,[1] by analyzing the optimality of an energy-state model, explicitly showed some mechanisms for enhanced cruise performance by periodic cruise.

The energy-state model is an approximation to the point-mass vehicle model by neglecting the altitude and flight-path angle dynamics. In the energy-state model, energy and fuel mass are state variables, thrust and velocity (or altitude) are considered control variables, and range is the independent variable. The hodograph is formed by determining the boundary of reachable rates of the state variables for admissible values of the control variables at a given energy. The steady-state cruise fuel performance is given by the value of the fuel-mass rate at the point at which the hodograph crosses the zero-energy-rate axis. If the hodograph is not convex so that a straight line tangent to two points on the hodograph (called the convex hull[2]) crosses the zero-energy axis at a smaller value of the fuel-mass rate than does the hodograph itself, then the control variables at the point of tangency are used to form a chattering control sequence, which implies the possibility of improved fuel performance over the steady-state cruise path. This chattering sequence, called the relaxed steady-state cruise[3] and first discussed by Edelbaum,[1] is an unrealizable infinite frequency control sequence between two operating points, one where the aircraft is aerodynamically efficient and the other where the aircraft is propulsion efficient.

Because velocity and altitude chattering is unrealizable, altitude was added as a state variable[4] and thrust and flight-path angle are considered the control variables. Schultz and Zagalsky[4] applied the small-angle approximation to the flight-path angle, resulting in flight-path angle and thrust appearing linearly in the aircraft dynamic model and cost function. These control variables, which lie interior to their admissible control sets along the extremizing steady-state cruise path, form what is called a doubly singular arc in the calculus of variations.[5,6] By applications of the Goh–Robbins[7,8] generalized Legendre–Clebsch condition, which test the convexity of the Hamiltonian in the optimal control problem, it has been demonstrated[9] that the steady-state cruise path is not minimizing.

The steady-state cruise arc using the aircraft model of Schultz and Zagalsky,[4] extended to the full point-mass model by including the flight-path angle as a state variable and the angle of attack as a control variable, has been shown[10] to satisfy the first-order necessary conditions and the generalized Legendre–Clebsch condition applied only to the thrust. However, this does not mean that steady-state cruise is a locally minimizing path in the calculus of variations. A second variational test must be applied. Because steady-state cruise is time- or range-invariant and is assumed to go on forever, the second variational accessory problem reduces to that of an infinite-time, quadratic cost problem subject to a time- or range-invariant linear dynamic system. By using Parseval's rule, the second variational accessory problem reduces to ensure the positive definiteness of a frequency-dependent matrix kernal.[11,12] This frequency-domain version of the Jacobi test fails[13,14] when applied to many aircraft models including that of Schultz,[10] and, therefore, steady-state cruise obtained from the full point mass aircraft model often is not a minimizing path.

Because steady-state cruise is not minimizing in many aircraft models, attention was drawn to determining the form and structure of the optimal periodic flight path by direct numerical computation. Gilbert and Lyons,[15] using a periodic spline parameterization of the state-control histories in conjunction with a nonlinear programming algorithm, and Speyer et al.,[16] using a shooting method, computed

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optimal oscillatory cruise trajectories. More realistic models were used later.[17,18] To produce these optimal periodic paths, the first- and second-order necessary conditions for optimality of periodic processes needed to be developed.

The initial studies of cyclic operation of engineering systems began with the work of Horn and Lin[19] in chemical processes. They first proposed that the periodic cost criterion be an average cost given as the value of the integral cost divided by the period of the oscillation. In this way the infinite time problem is reduced to a finite-time problem with periodic constraints on the state variables. The first-order necessary conditions for minimizing the average cost with respect to the controls, initial state, and period were similar to those of the classic calculus of variations, except that a new transversality condition resulted, which is associated with the optimality of the period of the process. The work in chemical processes concentrated on dynamic systems whose Hamiltonians were not convex in the control variables and, therefore, led to chattering or relaxed optimal control processes, and appeared to be of limited interest. However, because the frequency test predicted trajectories of finite period, second-order tests were required for testing the optimality of numerical periodic extremal paths.

Sufficient conditions for weak local optimality based on the second variation were first given by Bittanti et al.[20] for fixed period and by Chang[21] for free period. Unfortunately, some very essential issues that were well known in celestial mechanics[22,23] were not used by those researchers.[20,21] This is best understood by defining the monodromy matrix, which is the transition matrix determined from the linearized dynamics associated with the first-order necessary conditions over one period about a closed periodic trajectory. The monodromy matrix has two unity eigenvalues, which lie generically in the same Jordan box. This fact alone contradicts previous results[20,21] that required that there be no unity eigenvalues. Speyer and Evans[24] gave a sufficiency condition for a periodic process with free period to be a weak local minimum, taking into account the peculiarities of the periodic optimal path. Wang and Speyer[25] developed necessary and sufficient conditions for weak, local optimality of a periodic path, whereas only sufficiency was obtained by Speyer and Evans.[24] It is shown that for the single-period optimization, the Jacobi condition only requires that the solution to a matrix Riccati differential equation over one period exists starting at every point on the periodic path. However, if the number of orbits becomes infinite, the existence of a periodic solution to the Riccati differential equation is necessary for optimality. The infinitely repeated periodic process is an important basis for the development of the neighboring optimum periodic guidance law used to implement the optimal periodic path in the presence of initial condition and system uncertainties.[26]

This paper describes results in periodic optimal flight that have occurred over the past 25 years. To be somewhat complete, the general theoretical issues are also included. The interest of this paper is to place the author's work in perspective with that of other researchers and is not intended to be a survey of the field. In Sec. II, the periodic optimal flight problem is formulated and then generalized to allow a more succinct presentation of the theoretical results. In Sec. III, the simplest problem that generates optimal periodic paths, called the sailboat problem, is used to illustrate some of the properties of the optimal periodic solution. A particularly elegant aspect of this problem is the perturbation expansion away from the chattering solution. In Sec. IV, some of the mechanisms for periodic optimal flight are described and put in perspective. These mechanisms are essentially intuitive, having been extracted from reduced-order models such as the energy-state model or from a second variation analysis about the steady-state path. Nevertheless, from these fragments, some guidance for understanding the more complex optimal periodic path is obtained. In Sec. V, a hypersonic vehicle is described, based on an abstraction of the NASA generic hypersonic model,[27] and the periodic optimal flight path is obtained. In Sec. VI, the periodic neighboring optimum regulator is described and used to implement the periodic flight path. A key assumption is that, over one cycle, the mass change is negligible compared to the total weight. However, explicitly slow parameter changes can be included in the periodic optimal guidance rule. Finally, in Sec. VII, we conclude with a discussion of other applications for periodic optimal flight.

II. Formulation of Periodic Optimal Flight

The point-mass equations of motion of an aircraft flying in a vertical plane over a spherical nonrotating Earth are

$$h' = \tan\gamma[1 + (h/R_0)] \equiv f_h \tag{1}$$

$$M' = \left\{\frac{T - D - W\sin\gamma}{Ma^2W\cos\gamma}\right\}\left(1 + \frac{h}{R_0}\right) \equiv f_M \tag{2}$$

$$\gamma' = \left\{\frac{g(L - W\cos\gamma)}{M^2a^2W\cos\gamma} + \frac{1}{R_0 + h}\right\}\left(1 + \frac{h}{R_0}\right) \equiv f_\gamma \tag{3}$$

where $(\cdot)'$ denotes the derivative with respect to r, $\mathrm{d}(\cdot)/\mathrm{d}r$. The states of the system are altitude h, Mach number M, and flight-path angle γ. The independent variable is the range r. The system parameters are radius of the Earth R_0, speed of sound a, and vehicle weight W. Velocity $V = Ma$ also is used in subsequent discussion. The state variables h, V, γ, and forces are shown in Fig. 1.

The lift and drag forces are given as

$$L = \tfrac{1}{2}\rho a^2M^2S_bC_L, \qquad D = \tfrac{1}{2}\rho a^2M^2S_bC_D \tag{4}$$

where ρ is the atmospheric density, S_b is the reference area, and C_L and C_D are the lift and drag coefficients. The thrust T and C_L are considered the control variables, although engine thrust is related to throttle setting. The thrust is bounded as

$$T_{\min}(h, M) \le T \le T_{\max}(h, M, C_L) \tag{5}$$

where $T_{\min}$ and $T_{\max}$ are the thrust bounds.

The cost criterion for cruise is the mass of fuel used over one cycle $f(r_f)$ over the range of that cycle r_f, i.e.,

$$J = \frac{f(r_f)}{r_f} \tag{6}$$

where

$$f(r_f) = \int_0^{r_f} \frac{\sigma T}{Ma\cos\gamma}\left(1 + \frac{h}{R_0}\right)\mathrm{d}r \tag{7}$$

where σ is the thrust-specific fuel consumption. In this work and much of that in the literature to date, a linear dependency of fuel-mass rate on thrust is assumed.

A similar performance index can be obtained for endurance where time rather than range is the independent variable. Even though the periodic performance for endurance has been most dramatic,[28] the cruise problem, because of its inherent importance, is the focus of our analysis and simulation.

The fuel optimal cruise problem is to find the control variables $C_L(\cdot)$ and $T(\cdot)$, the initial conditions $h(0)$, $M(0)$, and $\gamma(0)$, and the period r_f that minimizes Eq. (6) subject to the differential equations (1), (2), and (3), the inequality constraint (5), and periodicity constraints

$$h(0) = h(r_f), \qquad M(0) = M(r_f), \qquad \gamma(0) = \gamma(r_f) \tag{8}$$

It is assumed that the weight change over one cycle is negligible and, therefore, W is a constant. This restriction is somewhat removed in the development of the periodic regulator discussed in Sec. VI.

A. General Formulation of Optimal Periodic Control Problem

The theoretical issues are more conveniently presented by formulating the above periodic optimal control in a general format. Consider that the state vector $x \in \mathcal{R}^n$, which could be composed of the elements (h, M, γ), is propagated by the dynamic system

$$\dot{x} = f(x, u) \tag{9}$$

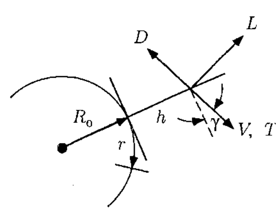

Fig. 1 States and forces for a point mass.

where the differential $\dot{x}$ is with respect to any independent variable such as range and time and $u \in \mathcal{R}^p$ is the control vector. Note that Eq. (9) represents an autonomous system, i.e., $f(x, u)$ is explicitly independent of the independent variable. For notational consistency with the literature, the independent variable is t.

The optimal periodic control problem consists of minimizing the performance criterion

$$J[u(\cdot), x(0), \tau] = \frac{1}{\tau}\int_o^\tau L[x(t), u(t)]\,dt \tag{10}$$

with respect to the period $\tau \in \mathcal{T} = [0, \infty)$, the p-vector control functions $u(\cdot) \in \mathcal{U} = \{u(\cdot)$: $u(\cdot)$ is piecewise continuous in the interval $[0, \infty)$ and $\|u(\cdot)\|_\infty \triangleq \sup_{t\in[0,\infty)} |u(t)| < \infty$, where $|u(t)| \triangleq [\sum_{i=1}^p u_i^2(t)]^{1/2}, u(t)\epsilon\mathcal{R}^p\}$, and the initial conditions $x(0) \in \mathcal{R}^n$, subject to the time-invariant dynamic constraint (9) and the periodic boundary conditions $x(\tau) = x(0)$. The assumption is made that $f(\cdot, \cdot)$ and $L(\cdot, \cdot)$ and their first and second derivatives are continuous with respect to both arguments.

The first-order necessary conditions for a weak local periodic minimum are stated with respect to the variational Hamiltonian, defined as

$$H(x, u, \lambda) = L(x, u) + \lambda^T f(x, u) \tag{11}$$

where $\lambda(t) \in \mathcal{R}^n$ is a Lagrange multiplier vector. It is assumed in the following statement of the minimum principle for an optimal periodic process that the optimization problem is normal.[29]

Proposition 1. A necessary condition for $[u^\circ(\cdot), x^\circ(0), \tau^\circ] \in \mathcal{U} \times \mathcal{R}^n \times \mathcal{T}$ being optimal for the above periodic optimal control problem is that there exist Lagrange multipliers λ such that (subscripted functions denote partial differentiation)

$$\dot{x}^\circ = f(x^\circ, u^\circ), \qquad \dot{\lambda} = -H_x^T(x^\circ, u^\circ, \lambda)$$
$$H_u(x^\circ, u^\circ, \lambda) = 0 \tag{12}$$

and

$$x^\circ(\tau) = x^\circ(0), \qquad \lambda(\tau) = \lambda(0), \qquad H = J \tag{13}$$

The periodicity of the Lagrange multipliers is required for optimality of the initial conditions, and the transversality condition $H = J$ is the optimality condition for the period τ° (Ref. 19).

B. Second Variation and Some Characteristics of the Periodic Optimal Control Problem

Important properties of the Hamiltonian system (12) and (13) are reviewed. In particular, the properties of the monodromy matrix, defined as the transition matrix evaluated over one period of the associated linearized variational equations, are presented and the necessary and sufficient conditions for optimality are expressed in terms of the associated solution to the Riccati differential equation. Let the state and Lagrange multiplier vector be defined as

$$y^T(t) \triangleq [x^T(t), \quad \lambda^T(t)] \tag{14}$$

By using the implicit-function theorem assuming that H_{uu} is positive definite, $H_u = 0$ implies that $u^\circ = \bar{u}(x, \lambda)$. Note that $H_{uu} \geq cI, c > 0$ is the strong form of the Legendre–Clebsch condition. The variational equations (12) become

$$\dot{y} = KH_y[y, \bar{u}(y)] \tag{15}$$

where K is the $2n \times 2n$ fundamental simplectic matrix

$$K = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \tag{16}$$

The linearized perturbation equation associated with Eq. (15) is simply

$$\delta\dot{y} = KH_{yy}\delta y \tag{17}$$

where $\delta y(t) = y(t) - y^\circ(t)$, in which the norm of δy is assumed sufficiently small so that first-order terms in a Taylor expansion dominate, and KH_{yy} is the $2n \times 2n$ Hamiltonian matrix.

The transition matrix associated with Eq. (17) and generated by

$$\dot{\Phi}(t, 0) = KH_{yy}\Phi(t, 0), \qquad \Phi(0, 0) = I \tag{18}$$

when evaluated at the end of one period τ is called the monodromy matrix $\Phi(\tau, 0)$. A well-known property of $\Phi(\tau, 0)$ is that it is symplectic. This characterizes the eigenvalues λ_i as $\lambda_i = 1/\lambda_{i+n}, i = 1, \ldots, n$. These eigenvalues are invariant with respect to the starting time or the starting point for the generation of the monodromy matrix $\Phi(\tau + t, t)$. An especially important property of $\Phi(\tau, 0)$ is that the velocity vector $\dot{y}(\tau)$ is an eigenvector with associated eigenvalue of unity. Furthermore, because $\Phi(\tau, 0)$ is symplectic, there is another unity eigenvalue, and from the perturbation equations of the periodic boundary conditions, it is seen that the associated eigenvector is a generalized eigenvector and the two unity eigenvalues are generically in the same Jordan box.[24]

Because H_y is autonomous, the differential equation describing the propagation of the velocity $\dot{y}$ is $\ddot{y} = KH_{yy}\dot{y}$ and has a solution on the periodic optimal path $\dot{y}(\tau) = \Phi(\tau, 0)\dot{y}(0)$. From the periodicity conditions (13), $\dot{y}(t) = \dot{y}(t + \tau)$ and for $t = 0$, we see that $\dot{y}(0)$ is an eigenvector of $[\Phi(\tau, 0) - I]\dot{y}(0) = 0$ with unity eigenvalue. From the linearized periodicity conditions (13), $\delta y(0) = dy(\tau) = \delta y(\tau) + \dot{y}(\tau)\,d\tau$ and by using the transition matrix property $\delta y(0) - \dot{y}(\tau)\,d\tau = \Phi(\tau, 0)\delta y(0)$, the relationship between the primary eigenvector $\dot{y}(0)$ and the generalized eigenvector $\delta y(0)$ is

$$[\Phi(\tau, 0) - I]\delta y(0) = -\dot{y}(0)\,d\tau \tag{19}$$

such that $[\Phi(\tau, 0) - I]^2\delta y(0) = 0$. If $d\tau \neq 0$, then the unity eigenvalues are in the same Jordan box. In Ref. 24, the generalized eigenvector is a direction associated with a one-dimensional family of solutions to the variational equation in which the index parameter is the Hamiltonian H. Small changes in $y(0)$ along the generalized eigenvector will produce another closed orbit with a small change in H. Note that H is a constant of the motion along any family member. The desirable H is when $H = J$, as given in Eq. (13). Note that for each change in H there is a change in τ, i.e., $\tau = \tau(H)$. The condition $\partial\tau/\partial H \neq 0$ ensures that the unity eigenvalues lie in the same Jordan box. Although this generic characterization of the periodic optimal control problem was missed previously,[20,21] the requirement that there be no unity eigenvalue was addressed and included in the conditions of Speyer and Evans.[24]

C. Necessary and Sufficient Conditions for Weak Local Optimality of Periodic Paths

The results of Speyer and Evans[24] corrected the previous sufficiency conditions for weak, local optimality. However, the gap between necessity and sufficiency was still unknown. An important result given by Wang and Speyer[25] was the clarification between the necessary and sufficient condition for a single-period orbit and those for an infinitely repeated orbit. The sufficiency conditions derived for a single orbit[24] were too restrictive for the single orbit, but consistent with those found for the infinitely repeated process.[25] This means that by using the results of Wang and Speyer,[25] single-period orbits that pass the test for sufficiency may not pass the tests for the infinitely repeated orbit. It is fortunate that the sufficiency tests of Speyer and Evans[24] that were obtained for the single orbit, a case that is used for convenience, is also the appropriate conditions for the infinitely repeated orbit that is the real focus of the periodic optimal control problem.

The classical Jacobi condition that tests for the existence of a conjugate point is closely related to the existence of a continuous, real, symmetric solution to the Riccati differential equation

$$\dot{P} = -PA - A^TP + PBP - C \tag{20}$$

where the $n \times n$ periodic coefficients A, B, C are determined from the Hamiltonian matrix evaluated on the periodic orbit as

$$\begin{bmatrix} A & -B \\ -C & -A^T \end{bmatrix} = KH_{yy} \tag{21}$$

The behavior of the Riccati equation is related to the partitioned $n \times n$ blocks of the $2n \times 2n$ transition matrix $\Phi(t, 0)$. The results of Wang and Speyer[25] elegantly relate the conditions for optimality of the control variable u° to the existence of the solution to the Riccati equation and the conditions for optimality of $x^\circ(0)$ and τ° to the semipositive definiteness of a $n+1 \times n+1$ matrix denoted $\bar{M}(t_o, t_o+\tau)$.

The necessary and sufficient conditions for weak local optimality of a single-period process[25] will now be stated and the differences with previous conditions[24] are explained.

Sufficient Condition. A second-order sufficient condition for an extremal periodic path to be a weak local minimum is that

1) For all $t_o \in [0, \tau)$, there exists a continuous real symmetric solution to the Riccati differential equation (20) on $t \in [t_o, t_o+\tau]$

2) For all $t_o \in [0, \tau)$ and $n+1$-vectors ξ, $\xi^T \bar{M}(t_o+\tau)\xi \geq 0$, and the equality is true only if $\xi = [\varepsilon f^T(t_o), 0]^T$, where ε is a real number. $\bar{M}$ is defined in Eq. (23).

Necessary Condition. Same statements as the sufficient condition, except that condition 2 is weakened as: For all $t_o \in (0, \tau)$, $\bar{M}(t_o, t_o+\tau) \geq 0$.

Condition 1 is a conjugate point condition associated with the optimality of the control $u(\cdot)$. If condition 1 is satisfied, the second variation of the cost $\delta^2 J$ becomes

$$\delta^2 J = \frac{1}{\tau}\left[\delta x(t_o)^T, \, \mathrm{d}\tau\right]\bar{M}(t_o, t_o+\tau)\begin{bmatrix}\delta x(t_o) \\ \mathrm{d}\tau\end{bmatrix} \tag{22}$$

where $\bar{M}(t_o, t_o+\tau)$ is made up of partitioned $n \times n$ blocks of the monodromy matrix $\Phi(t_o+\tau, t_o)$ as

$$\bar{M}(t_o, t_o+\tau) = \begin{bmatrix}\Phi_{12}^{-1}(I-\Phi_{11}) + \Phi_{12}^{-T}\left(I - \Phi_{22}^T\right), & \tilde{H}_x^T + (\Phi_{22} - I)\Phi_{12}^{-1}\tilde{f} \\ \tilde{H}_x + \tilde{f}^T\Phi_{12}^{-T}\left(\Phi_{22}^T - I\right), & -\tilde{H}_x\tilde{f} - \tilde{f}^T\Phi_{22}\Phi_{12}^{-1}\tilde{f}\end{bmatrix}_{t=t_o+\tau} \tag{23}$$

where Φ_{ij}, $i, j = 1, 2$ are partitioned blocks of $\Phi(t_o+\tau, t_o)$ and $\tilde{H}_x$ and $\tilde{f}$ are H_x and f evaluated at $t = t_o+\tau$. Condition 1 ensures that Φ_{12} is invertible. State perturbations in the direction of $f(t_o)$ do not change the orbit, and therefore have no effect on the cost functional. There is no requirement that the Riccati equation be periodic, as there is in earlier works.[20,21,24] Furthermore, the restriction that the eigenvalues of the monodromy matrix be distinct except for two unity eigenvalues, which were required by Speyer and Evans[24] for the proof of the strongly positive optimality condition, is no longer needed in later work.[25]

Although the conditions for optimality for the single-period orbit have been clarified,[25] it is the infinitely repeated periodic process that is of central concern. Again, Wang and Speyer[25] were able to clarify the necessary and sufficient conditions. These conditions are closer to those obtained earlier[24] and are the following: The necessary condition for local weak optimality of an infinitely repeated periodic process is that there exists a continuous real symmetric periodic solution to the Riccati differential equation (20); the sufficient condition is the same as necessity but adds the required condition that the monodromy matrix has no eigenvalues on the unit circle except for the pair of unit eigenvalues. Note that the gap between necessity and sufficiency is minimal. The necessary and sufficient conditions for the infinitely repeated process are used to test optimality in the forthcoming examples.

D. Additional Optimality Conditions Required for the Hypersonic Cruise

There are two important extensions that are required for our results in Sec. V. The first is that the thrust control enters linearly into the dynamics and cost criterion in Sec. II. The Pontryagin minimum principle is used to determine the optimality of the thrust T or throttle setting S as used in Sec. V. Because singular arcs are not anticipated, the thrust is assumed bang-bang where the switch points are determined by minimizing the Hamiltonian. No limit to the number of switches is made. If the switches accumulate on certain arcs, then singular control is possible, and must be explicitly determined. It is assumed that singular arcs will not occur, and this assumption is verified by our results. Because the optimization problem is formulated from an autonomous dynamic system and cost criterion, the Hamiltonian is shown to be a constant of the motion.

The local optimality of the thrust control sequence is difficult to determine because the test for local optimality of the thrust sequence requires strong variations in the thrust control. To circumvent the difficulty of producing a field of extremals required for sufficiency, the second variation test is placed on the optimality of the switch times (ranges) rather than the thrust control function.[30,31] This requires the assumption that the number of switches do not change with variations in the switch times. At the switch times, the Pontryagin minimum principle is equivalent to the stationary condition that the gradient of the cost criterion with respect to the switch times is zero.[16] As given elsewhere,[16,26,30–32] a sufficiency condition for local optimality is that the second partial of the cost criterion with respect to the switching time is positive definite. This approach forms the basis of the regulator where changes in switch times are linear functions of the perturbations in the state away from the nominal periodic path. This approach poses no contradiction to the autonomy of the first variations with respect to the independent variable.

The second extension involves the inclusion of a state and control inequality constraint. In particular, in Sec. V, the trajectory is required to satisfy a g-loading inequality constaint. As shown elsewhere,[32] this constraint is added to the variational Hamiltonian by a Lagrange multiplier which is zero when the trajectory is not on the constraint and nonnegative when it is on the constaint manifold. The second variation, again, requires modification. In particular, the Riccati equation (20) is modified on the constraint boundary where satisfaction of the constraint determines some of the controls. If on certain arcs, all of the controls are required to satisfy the constraints,[30–32] then on those arcs the Riccati equation reduces to a Lyapunov equation. All first- and second-order necessary and sufficiency conditions required for the hypersonic periodic cruise problem formulated in Sec. V are summarized elsewhere.[32]

Note that at points of control discontinuity, the solution to the Riccati equation also can be discontinuous.[30,31] Dewell and Speyer[32] computed the transition matrix for the perturbed Hamiltonian system, from which the solution to the Riccati equation can be derived. The jumps in the transition matrix are determined at switch times, and it is verified that the transition matrix retains its symplectic property. In Dewell and Speyer,[32] the jumps in the throttle setting are attributable either to satisfying the optimality of the Hamiltonian or satisfying the inequality constraint at the junction to the boundary. These second-variation results are the basis for the neighboring optimum regulator presented in Sec. V and explicitly developed earlier,[32,33] where variations in lift coefficient and the switch times are related to the perturbations in the state from the nominal optimal periodic path.

III. Illustration of Optimal Periodic Control: Sailboat Problem

With the theory for periodic optimal processes in a reasonable state of development, illustrative problems are sought that are simple enough to allow intuitive insight into the periodic phenomena, but complex enough to induce optimal periodic processes. Autonomous optimal control problems subject to a scalar differential constraint do not admit periodic optimal processes. Because the Hamiltonian is a constant of the motion, this class of optimization problems reduces to the solution of a scalar differential equation, which does not admit periodic solutions. This class is also integrable in the sense that the optimal control problem reduces to quadrature integrations. It appears that no integrable optimal control problems admit a periodic optimal control solution. Therefore, no analytical illustrations are available.

The following periodic optimal control problem appears to be the simplest obtainable. Its solution is expressed through numerical simulation and asymptotic approximations. The asymptotic approximations are of interest in themselves, representing interesting multiple time-scale asymptotic solutions. The optimal periodic problem is to find the scalar control $u(\cdot)$, the period τ, and the vector of initial states, $x(0)^T = [x_1(0), x_2(0)]$, that minimize the performance index

$$J = \frac{1}{\tau}\int_o^\tau \left(\frac{x_1^2}{4} + \frac{x_2^4}{4} - \frac{x_2^2}{2} + \frac{bu^2}{2}\right) dt \tag{24}$$

subject to the second-order dynamic constraints

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = u \tag{25}$$

and to the periodicity conditions

$$x_1(0) = x_1(\tau), \qquad x_2(0) = x_2(\tau) \tag{26}$$

where b is a weighting parameter on the control. Because a nonconvex cost seems to induce periodic optimal paths, a negative quadratic term is included in Eq. (24). As shown, the weighting parameter b determines whether the optimal solution is a periodic or a static path. Furthermore, note that if $b = 0$, x_2 is considered the control, and the lack of convexity is moved from the state variable to the control variable. Nonzero b will induce oscillations with finite frequency.

The problem might physically represent a sailboat attempting to maximize its average velocity into the direction of the wind. Suppose z is the distance in the direction of the prevailing wind, and x_1, x_2 in Eq. (25) represent the lateral position and velocity, respectively. Then, the average velocity

$$\frac{-z(\tau) + z(0)}{\tau} = \int_o^\tau -\frac{\dot{z}\,dt}{\tau} = \frac{1}{\tau}\int_o^\tau \left(\frac{x_2^2}{2} - \frac{x_2^4}{4}\right) dt$$

is to be minimized subject to an integral penalty on the lateral position $\int_o^\tau (x_1^2/2)dt/\tau$ associated with the tack of the sailboat, and the cost of tacking

$$\int_o^\tau \left(\frac{bu^2}{2}\right)\frac{dt}{\tau}$$

The longitudinal velocity $\dot{z}$ is assumed to be a function of the lateral velocity x_2.

The variational Hamiltonian defined in Eq. (11) becomes

$$H = (x_1^2/2) + (x_2^4/4) - (x_2^2/2) + (bu^2/2) + \lambda_1 x_2 + \lambda_2 u \tag{27}$$

where λ_1 and λ_2 are Lagrange multipliers associated with Eq. (25). Let $y^T \triangleq [x_1, x_2, \lambda_1, \lambda_2]$, and note that $H_{uu} = b \geq 0$. Then, the first-order necessary conditions become

$$\begin{aligned} \dot{x}_1 &= x_2, & \dot{x}_2 &= -\lambda_2/b \\ \dot{\lambda}_1 &= -x_1, & \dot{\lambda}_2 &= x_2 - x_2^3 - \lambda_1 \end{aligned} \tag{28}$$

where the periodicity conditions are $y(0) = y(\tau)$ and the transversality condition is $H = J$. Note that $H_u = 0$ implies $u = -\lambda_2/b$ and was used to derive Eq. (28).

In the following asymptotic analysis, b plays a central role. First, we determine the range of values of b in which the steady-state solution to the first-order necessary conditions are no longer minimizing. The second variation about the static extremal using Parseval's relation[11,12] is

$$2\delta^2 J = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \delta u(-i\omega)\pi(\omega)\delta u(i\omega)\, d\omega \geq 0 \tag{29}$$

in which optimality implies that the kernal $\pi(\omega) \geq 0$. This is called the π-test, and application to this problem requires that $\pi(\omega) = 1/\omega^4 - 1/\omega^2 + b \geq 0$ for optimality. The minimum value of $\pi(\omega)$, independent of b, occurs when $\omega = \sqrt{2}$. Therefore, for

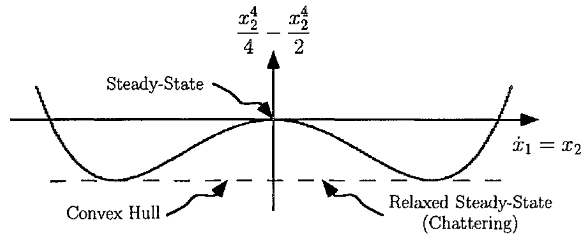

Fig. 2 Hodograph for the sailboat problem when $b = 0$ and $x_1 = 0$.

$0 \leq b < \frac{1}{4}$, there is a frequency range for which $\pi(\omega) < 0$, and the static path is not minimizing. For $b \geq \frac{1}{4}$ the static path is minimizing. Therefore, the problem is constructed so that for $0 \leq b < \frac{1}{4}$, periodic optimal paths that are minimizing can be obtained. Although numerical periodic optimal paths have been obtained,[24,34] interesting analytical asymptotic expansions are obtained near $b = \frac{1}{4}$ and $b = 0$.

For the case $b = (1 - \bar{\varepsilon}^2)/4$, where $\bar{\varepsilon}$ is a small expansion parameter, a regular perturbation scheme is developed by using the Lindstedt–Poincaré expansion method.[35] The static path is used as a reference path to expand the solution. The resulting expansion is essentially a harmonic series in which, for small values of the expansion parameter and a few terms of the series, excellent agreement with the numerical solution is obtained. For example, the expansion for the period and the cost criterion are $\tau = \pi 2^{1/2}(1 - \bar{\varepsilon}^2/4 - 91\bar{\varepsilon}^4/1024 + \cdots)$ and $J = \bar{\varepsilon}^4/24 - 17\bar{\varepsilon}^6/768 + \cdots$, respectively. Given the symmetry conditions assumed for this problem, it is supposed that there is only one optimal path. The results from this asymptotic analysis are somewhat expected.

A more dramatic asymptotic analysis is about $H_{uu} \equiv b = 0$.[36] The chattering optimal solution with an infinite frequency in x_2 is used as the reference solution for the asymptotic expansion. If the penalty on u is removed ($b = 0$) and x_2 becomes the control, then an infinite chattering between the peaks of $\dot{z}$ at $x_2 = \pm 1$ produces the maximum average velocity along z, with no lateral excursions in x_1. This problem can be relaxed by taking the convex hull of the hodograph, the boundary of reachable values of $\dot{z}$ and $\dot{x}_1$, with respect to x_2. The convex hull is a line tangent to $\dot{z}(x_2)$ at $x_2 = \pm 1$, as shown in Fig. 2. This phenomenon attributable to the reduction in the state space will occur again in the analysis of the point-mass flight model.

For this simple and somewhat transparent periodic control problem, the neglected dynamics $\dot{x}_2 = u$ at $b = 0$ are included by an asymptotic expansion about the chattering solution. By using b as the expansion parameter, two timescales are developed for the asymptotic expansion. One timescale, proportional to the period, is used to transform the problem to one similar to that of a relaxation oscillator, where the problem is characterized by slow, almost equilibrium, motion near the peaks at either $x_2 = \pm 1$ connected by fast, jump-type transitions. The asymptotic expansion is divided into two parts, an outer part at the timescale of the period and an inner part characterized by an even faster timescale that captures the fast transitions. These two solutions are matched together to obtain the resulting asymptotic solution.

In the outer region the new timescale is $\tau_1 = t/b^{1/6}$, where, as in the inner region or boundary layer, the timescale is $\tau_2 = t/b^{1/2}$. Once expansions of the state and Lagrange multipliers are made in both the outer and inner regions using these very special timescales, the matching of the outer with the inner solution is developed by a principle proposed by Van Dyke.[37] Once the matching has been completed, asymptotic expansions for the period and the cost are obtained as

$$\tau = 4b^{\frac{1}{6}}\left(2^{\frac{1}{6}} + \frac{b^{\frac{1}{3}}2 \cdot 2^{\frac{3}{6}}}{15} + \frac{b^{\frac{2}{3}}463 \cdot 2^{\frac{5}{6}}}{6300} + \cdots\right) \tag{30}$$

$$J = -\tfrac{1}{4} + b^{\frac{1}{3}}\left(1/2^{\frac{2}{3}}\right) - b^{\frac{2}{3}}\left(2^{\frac{2}{3}}/30\right) - b[(\pi^2/12) + (117/6300)] + \cdots \tag{31}$$

A sketch of the lateral position, velocity, and control over one cycle is shown in Fig. 3, where the order of the magnitude of the control is given in the inner and outer regions.

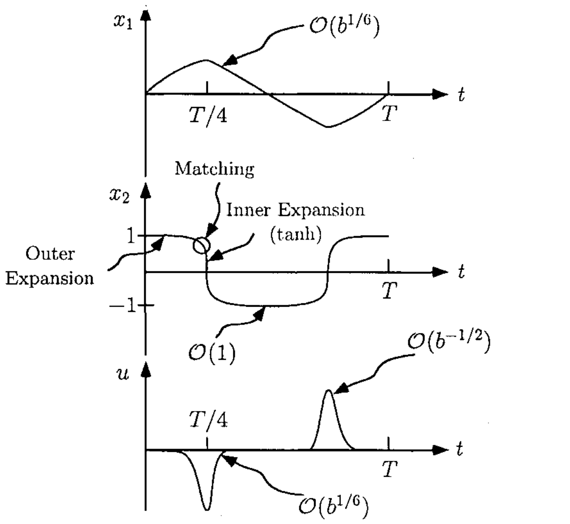

Fig. 3 Lateral position (x_1), velocity (x_2), and control (u) over one cycle for the sailboat problem.

The characterization of the solution of this simple optimal periodic control problem about the chattering solution as a relaxed oscillator gives some clues as to the possible behavior of the periodic optimal flight problem.

IV. Mechanisms for Periodic Optimal Flight

Analysis of periodic fuel optimal cruise began with the reduced state order energy-height model. The energy-height variable is $E = (1/2g)V^2 + h$. Using the velocity rather than M, and somewhat simplifying the dynamics in Eqs. (1–3),

$$\frac{\mathrm{d}E}{\mathrm{d}r} = \frac{(T-D)}{W} \tag{32}$$

The altitude and flight-path dynamics are assumed neglectable, so that $L = W$. The fuel rate per range is $\mathrm{d}f/\mathrm{d}r = \sigma T/V$. The system dynamics involve only the states E and f, where the controls are now the thrust T and the velocity. The hodograph at constant E (f does not enter the right side of the equations) is the reachable values of the rates $\mathrm{d}E/\mathrm{d}r$ and $\mathrm{d}f/\mathrm{d}r$ for admissible values of the controls (T, V). Only the upper boundary of the hodograph is shown in Fig. 4. The hodograph may be constructed by choosing a value of $\mathrm{d}E/\mathrm{d}r$, where $T = W\mathrm{d}E/\mathrm{d}r + D$ and finding the smallest value of $\mathrm{d}f/\mathrm{d}r = \sigma(W\mathrm{d}E/\mathrm{d}r + D)/V$ with respect to V for bounds on T given by Eq. (5). The cruise point is where the hodograph crosses the axis $\mathrm{d}E/\mathrm{d}r = 0$. The chattering cruise point is obtained by taking the convex hull as indicated by the dashed line that is tangent to the hodograph from the engine off point. Chattering cruise[2] averages the vehicle performance between the flight conditions where the vehicle is propulsion efficient (maximum thrust) and aerodynamically (maximum L/D) efficient.

The effect of the lack of convexity of the hodograph on the optimization problem can be obtained by first constructing the variational Hamiltonian for this reduced state problem as

$$H = \frac{\sigma T}{V} + \frac{\lambda_E(T-D)}{W} \tag{33}$$

where λ_E is the Lagrange multiplier associated with the energy. First note that the first-order necessary conditions are satisfied along the static cruise path. From $H_T = 0, \lambda_E = -W\sigma/V$ and $H_V = (\sigma/V)_V T - \lambda_E D_V/W = 0$ combine, using the static cruise thrust $T = D$, as $(\sigma D/V)_V = 0$. Note that this implies the usual conditions for a static optimal cruise path with respect to h and V because here, h depends on V through the energy-height definition, i.e., $(\sigma D/V)_V$ is a total derivative such that

$$\left(\frac{\sigma D}{V}\right)_V = \left[\frac{\partial(\sigma D/V)}{\partial h}\right]\frac{\partial h}{\partial V} + \left[\frac{\partial(\sigma D/V)}{\partial V}\right] = 0$$

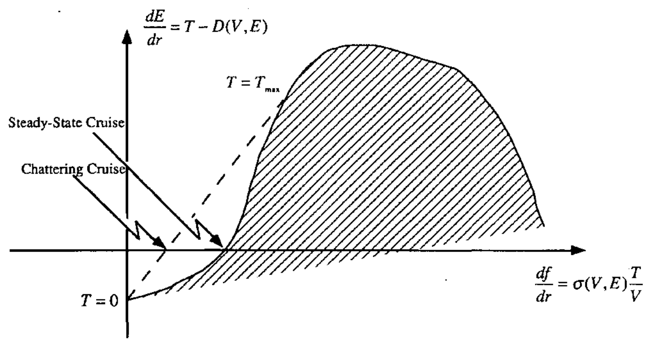

Fig. 4 Hodograph for an aircraft: $D \sim$ drag, $\sigma \sim$ thrust specific fuel consumption, and $E = (V^2/2g + h)$ = const.

The Legendre–Clebsch condition evaluated at the cruise point, where $T = D$ is not on a thrust bound [Eq. (5)], must be nonnegative definite, i.e.,

$$\begin{bmatrix} H_{TT} & H_{TV} \\ H_{VT} & H_{VV} \end{bmatrix} = \begin{bmatrix} 0 & (\sigma/V)_V \\ (\sigma/V)_V & H_{VV} \end{bmatrix} \geq 0 \tag{34}$$

However, for the above aircraft model this can be true only if $(\sigma/V)_V = 0$, otherwise, the matrix is always indefinite. This particular case occurs when the thrust-specific fuel consumption is modeled as $\sigma = \sigma_o V$, where σ_o is a constant. This case is of some theoretical interest because it indicates that chattering cruise will not improve static cruise performance. If periodic optimal cruise does improve fuel performance for this case, then there are other mechanisms attributable to the system dynamics that enhance periodic cruise. In one case,[38] thrust-specific fuel consumption for propeller propulsion appears to fit this model well. For general $\sigma(h, v)$, Eq. (34) is not satisfied, and chattering cruise improves performance. Furthermore, it is noted[2] that chattering cruise performance can be enhanced by increasing the ratio of maximum thrust to minimum drag as indicated on the hodograph of Fig. 4.

The long-range aircraft cruise problem has been analyzed using a model intermediate in complexity between the energy-state model and the point-mass model. In this model, altitude and velocity are elevated to state variables, but the flight-path angle is now a control along with thrust; γ' is assumed to be small so that $L = W$ is still used. The dynamic equations used[4,39] are

$$V' = (T-D)/mV - g\gamma/V, \qquad h' = \gamma \tag{35}$$

If we suppose that, for static cruise, $T = D$ is not on the thrust bound and $\gamma = 0$, then the first-order necessary conditions derived from the variational Hamiltonian $H = \sigma T/V + \lambda_V[(T-D)/mV - (g\gamma/V)] + \lambda_h\gamma$ are satisfied. In particular, the conditions for static cruise, $(\sigma D/V)_V = (\sigma D/V)_h = 0$, are satisfied. The optimality of static cruise using the extremal controls $T = D$ and $\gamma = 0$ is difficult to determine because the controls enter linearly in the Hamiltonian and, therefore, the Legendre–Clebsch condition is satisfied trivially in weak form, i.e., if $u^T = [T, \gamma]$, then $H_{uu} \equiv 0$. Because of the singular nature of the Legendre–Clebsch conditions, the cruise path is referred to as a singular arc. Because both controls are linear and assume interior values in their constraint set, the generalized Legendre–Clebsch condition, attributable first to Kelley[40] and later generalized to the vector control case by Robbins[7] and Goh,[8] needs to be applied. By deriving an alternative form of the second-variation cost, it is shown[5] that by using special control variations such as doublets, the generalized form of the Legendre–Clebsch condition is necessary for optimality. An aspect of this condition for the vector case applied to test the optimality of the cruise controls of the intermediate model is

$$\begin{bmatrix} \left[\dfrac{\mathrm{d}(H_T)}{\mathrm{d}r}\right]_T & \left[\dfrac{\mathrm{d}(H_T)}{\mathrm{d}r}\right]_\gamma \\ \left[\dfrac{\mathrm{d}(H_\gamma)}{\mathrm{d}r}\right]_T & \left[\dfrac{\mathrm{d}(H_\gamma)}{\mathrm{d}r}\right]_\gamma \end{bmatrix} = \begin{bmatrix} 0 & \dfrac{G}{V} \\ -\dfrac{G}{V} & 0 \end{bmatrix} \tag{36}$$

where

$$G = g[\sigma_h/g - (\sigma/V)_V] \tag{37}$$

By necessity this skew-symmetric matrix must be zero, implying that $G = 0$. Therefore, in general, the cruise condition will not yield a minimizing singular arc.[9]

In a second-variation analysis of this intermediate model about the static cruise, a term involving G appears.[39] Defining δV, δh, and $\delta\gamma$ as variations in V, h, and γ from their nominal static cruise values, the second-variation $\delta^2 J$ can be written as

$$\delta^2 J = \int_o^{r_f} [\delta V \quad \delta h] \begin{bmatrix} \mu_{VV} & \mu_{Vh} \\ \mu_{hV} & \mu_{hh} \end{bmatrix} \begin{bmatrix} \delta V \\ \delta h \end{bmatrix} \mathrm{d}r - 2m \int_o^{r_f} G \delta V \delta\gamma \, \mathrm{d}r \tag{38}$$

where $\mu = \sigma D/V$. The first term is associated with the optimality of the static cruise and is therefore positive. The second term is difficult to access but, for the aircraft models chosen,[39] G is positive and for a fighter aircraft $\delta V \delta\gamma$ is also positive. The second term is therefore negative.

This term was partially obtained by Breakwell and Schoall[14] by using the more complex point-mass model. The variational Hamiltonian was expanded to second order and terms were identified that may be responsible for fuel savings along oscillatory cruise trajectories. Menon[39] showed that

$$-2m \int_o^{r_f} G \delta V \delta\gamma \, \mathrm{d}r = -m \int_o^{r_f} \left[\frac{\sigma_h - g\sigma_V}{V}\right] \delta V \delta\gamma \, \mathrm{d}r - 2\int_o^{r_f} \left(\frac{m\sigma g}{V^2}\right) \delta V \delta\gamma \, \mathrm{d}r \tag{39}$$

Because the last term is related to the second variation through the gravity term in the velocity equation (2), it is termed a gravity effect by Breakwell and Schoall.[14] This may be the dominant effect here because for many aircraft, σ_h is approximately equal to zero and σ_V is positive or approximately zero. For this term to induce additional fuel efficiency, the variations δV and $\delta\gamma$ must be in phase. It is then seen that the gravity effect is related to the control convexity conditions in the generalized Legendre–Clebsch condition obtained earlier.[9]

Breakwell and Schoall[14] identified another term in his second-variation analysis that is absent from the second-variation analysis of the intermediate model of Menon,[39] where $L \cong W$. This term is called the induced drag effect and has the form, in the integrand of the second variation, of $(\sigma D_{Lh}/V)\delta L \delta h$. Because D_{Lh} is always positive, the induced drag effect contributes if δL is out of phase with δh. The intuition gained by this analysis is applied to the numerical optimal periodic trajectories presented in the next section, and are shown to be substantially correct.

A special case is considered where there is no advantage of chattering cruise over steady-state cruise as viewed in the energy-height model. If the static path is not minimizing, then the mechanisms for producing periodic cruise must come from the dynamic behavior of the point-mass motion. In an earlier work,[13] the system parameters in the point-mass dynamics were simplified by choosing profile and induced drag coefficients as constants and the thrust-specific fuel consumption as $\sigma = \sigma_o V$ where σ_o is a constant. The cruise altitude and velocity are found to be any point on a constant dynamic pressure line obtained by minimizing the specific drag model. Note that with this model the Legendre–Clebsch condition of Eq. (34) and the generalized Legendre–Clebsch condition of Eq. (36) are satisfied because $\sigma_h = (\sigma/V)_V = 0$. By using the π-test form of the Jacobi condition, static cruise will not be minimizing if $(L/D)_{\max}/V < 2\sqrt{(2\beta/g)}$, where g is gravity acceleration and $1/\beta$ is the atmospheric scale height. If either the maximum $L/D[(L/D)_{\max}]$ is low or the velocity is high, a cyclic process will be minimizing. The strategy for increasing fuel efficiency is to properly modulate the interchange of potential and kinetic energy. Part of this strategy should be to reduce the average induced drag as indicated by the perturbed induced drag effect identified by Breakwell and Schoall,[14] the only surviving mechanism for this special case.

This special problem was pursued further by Sachs.[38] He notes that this problem is of practical significance because the fuel-consumption characteristics of a propeller-driven aircraft are similar to $\sigma = \sigma_o V$ and the speed range corresponds to the incompressible subsonic Mach number region, resulting in an approximate Mach-number independent lift-drag polar. Sachs shows that even though the static path is locally minimizing, there may be a periodic optimal path that produces lower cost than the static cruise path, i.e., the global optimum is periodic. The velocity bound is the result of determining when the static cruise path becomes nonminimizing. For some velocities below this velocity bound, the static cruise path is a local minimum, and passes all the tests for optimality. It is significant that the static cruise path may only be a local minimum for low velocities whereas the velocity bound given above implies reasonably high aircraft velocities. Furthermore, Sachs shows for this special class of aircraft that minimum average fuel per range, which is proportional to energy into the system is equivalent to minimum average drag per range, which in turn is proportional to energy out of the system. Sachs shows numerically that by appropriately modulating the vehicle states and control, the average drag per range can be made smaller than the minimum drag aerodynamically possible in the static cruise. The mechanism suggested is that the drag decrease associated with downward flight-path curvature is larger because of the higher speeds than the drag increase in the upward flight-path curvature attributable to the lower speeds. This mechanism seems to be related to the gravity effect of Breakwell and Schoall.[14]

All of these mechanisms are somewhat heuristic; nevertheless, many insights are gained. In the following section the numerical optimal periodic path of a hypersonic vehicle is obtained, and the validity of some of these mechanisms is assessed.

V. Numerical Computation of Periodic Optimal Cruise

Once it became evident that the static cruise path was not optimal, interest turned to determining those paths that are fuel efficient. Our approach has been to develop a second-order shooting method for determining optimal flight paths tailored to the periodic optimal control problem.[16,17,34] The objective is to obtain precisely the minimizing periodic trajectory to determine the percentage of improvement in fuel per range cost over the optimal static cruise path. Furthermore, these paths may be a guide to a better understanding of the underlying mechanisms that induce these paths.

A. Numerical Optimization Procedure

In the shooting method about a nominal path constructed on each iteration, the computation of the transition matrix based on the linearized equations of motion is used to determine changes in the initial conditions of both state and Lagrange multiplier variables attributable to desired first-order changes in the terminal conditions. To converge to a particular periodic path as the orbit is closed, the matrix $(\Phi - I)$ used to determine changes in initial conditions becomes singular as Φ converges to the monodromy matrix with coupled unity eigenvalues. However, by fixing one element of the orbit (say $\gamma = 0$) that removes one column of $(\Phi - I)$, the indeterminancy is removed.[34]

Our approach to numerical optimization is the following. The shooting method is initialized by an arbitrary choice of the initial states and multipliers. If this choice is close to some periodic path with nonminimizing period, then convergence toward a closed path with arbitrary period may be quite rapid. The converged periodic orbit lies on a one-dimensional family indexed by the Hamiltonian, a constant of the motion on any particular family member. Changes in the family direction are obtained by making small changes in y along the generalized eigenvector of the monodromy matrix. This vector is used to start the search for the optimal period along the one-dimensional family. As the number of points along the family are obtained, a polynomial fit of those points is used to accelerate the search, and thereby accelerate the convergence to the family member with the optimum period. Some variations on this procedure have been reported.[16,34]

In the vehicle model discussed in Sec. II, the thrust enters linearly in both the dynamics and performance index. Therefore, the thrust enters linearly in the Hamiltonian, and the Legendre–Clebsch condition is met only in weak form. Two control strategies for the thrust can exist. Either the solution is bang-bang in that the assumed thrust switches from one bound to the other at distinct points, or the thrust takes on values interior to its control bounds for a finite interval of range (or time); this is called a singular arc. In the numerical results to be discussed, there was no limit placed on the number of switch points. Because an infinite-chattering solution produces an identical value of the performance criterion as a control on an optimal singular arc,[6] if the number of switches begins to accumulate in a particular interval of the range, then a possible singular arc is implied numerically. In the particular aircraft models considered here, this phenomenon did not occur. However, for a different class of aircraft, optimal control processes with singular arcs have been reported.[41,42] (A shooting method was also used by Sachs et al.[42])

The switch function obtained by the Pontryagin minimum principle does not limit the number of switches. However, it is convenient mathematically in deriving the linearized prediction matrix in the shooting method to consider the switch times as control parameters rather than the thrust as a control function once the number of switches has been established.[16,17] The assumption in the algorithm is that a singular arc does not exist. If it does, then the algorithm must be modified. The switch range control parameters were not explicitly considered in the necessary and sufficient conditions discussed in Sec. II.C. However, the inclusion appears to be straightforward and is discussed in Sec. II.D.

B. Aircraft Models

Various point-mass models have been proposed of varying realism. Gilbert and Lyons,[15] using a fairly crude model, reasonably met the chattering cruise performance predictions by a point-mass model, and made significant improvements if an altitude constraint was imposed. The numerical optimization technique used a spline approximation to the control functions, which turned the function minimization into a parameter minimization problem. Accelerated gradient codes are used to determine the approximate optimal control sequence. This approach leads to smooth thrust histories rather than discontinuous sequences, thereby missing some of the fine detail of the solution. At about the same time, the shooting method was developed, and applied to a point-mass model over both a flat and spherical Earth[16] where the thrust bounds, as in Ref. 15, were constants. The motivation in Speyer et al.[16] was to explicitly show that periodic paths occur even in the case where the reduced models do not predict performance improvement. This is the case where the thrust-specific fuel consumption is proportional to velocity and the π-test indicates that there is a velocity in which the static cruise path becomes nonminimizing, as discussed in Sec. IV. Sachs[38] showed that periodic optimal paths exist below the velocity for nonminimizing static cruise. Nevertheless, it was felt that significant fuel savings could be obtained by very energetic vehicles. This motivated the choice of vehicle model used earlier.[16]

Fuel-optimal periodic cruise paths were determined numerically by Chuang and Speyer[17] for a hypersonic vehicle with more realistic aerodynamics and scramjet engine models than used by Speyer et al.[16] The model parameters were curve-fitted to data obtained from NASA Langley Research Center. It was found that the engine-off drag penalty eliminates the advantages of the periodic cruise over the steady-state cruise for this model, although if removed, a 5% increase in fuel performance results. Nevertheless, the resulting periodic path was shown to be a locally minimizing path by the second-order test given in Sec. II. Furthermore, Fig. 5 shows that the mechanisms deduced from the second-order analysis about the static cruise path by Breakwell and Schoall[14] essentially hold for the locally periodic optimal path. Because the Mach number hovered about 6, possibly even more energetic vehicles would produce better fuel savings. Dewell and Speyer[18,32] used a generic hypersonic vehicle model proposed by NASA Dryden Reseach Facility.[27] For a reasonable approximation of this model, as given below, the periodic optimal orbit yielded a 10.8% improvement in fuel consumption over static cruise where the engine-off drag penalty was included.

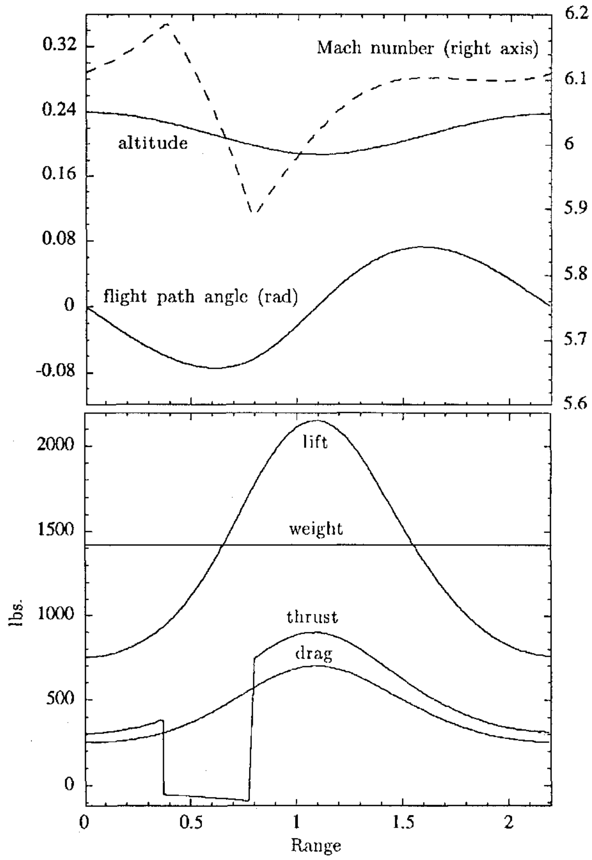

Fig. 5 Plots of optimal periodic Mach number, altitude (5×10^5), flight-path angles, lift, thrust, and drag vs range (5×10^5) using NASA Langley hypersonic vehicle data.

C. Generic Hypersonic Aircraft Model

The periodic optimal control problem formulated in Eqs. (1–8) is solved in slightly modified form by Dewell and Speyer[32] using the generic hypersonic vehicle proposed. To simplify the numerical computation, the atmospheric density is assumed to obey the exponential relation $\rho = \rho_o \exp(C_\rho h)$. The lift-drag polar is fitted to be linear in normalized velocity M as $C_D = d_{00} + d_{01}M + (d_{10} + d_{11}M)\alpha + (d_{20} + d_{21}M)\alpha^2$, where the lift coefficient at high velocities is reasonably approximated as $C_L = b_{00} + b_{01}\alpha + (b_{10} + b_{11}\alpha)M$, where the d and b are fitting parameters.[32]

The greatest uncertainty in the modeling process occurs in describing the engine. The engine thrust is modeled as a linear function of throttle setting as $T = ST_{\max} + (1 - S)T_{\min}$, where S is engine throttle setting and satisfied the constraint $0 \leq S \leq 1$. $T_{\max}$ and $T_{\min}$ are given in terms of their aerodynamics derivatives

$$T_{\max} = \tfrac{1}{2}\rho a^2 M^2 S_e C_{T_{\max}}, \qquad T_{\min} = -\tfrac{1}{2}\rho a^2 M^2 S_e C_{T_{\min}}$$

where $C_{T_{\max}}$ is a function of normalized velocity and angle of attack as

$$C_{T_{\max}} = C_{00} + C_{01}\alpha + (C_{10} + C_{11}\alpha)M + (C_{20} + C_{21}\alpha)M^2$$

where the C are fitting parameters[32] and $C_{T_{\min}}$ is assumed constant. The $T_{\min}$ is the drag penalty imposed on the vehicle by an additional axial drag force resulting from the airflow through the engines when the throttle setting is zero. The model parameters used are given elsewhere.[32] Because throttle setting is the control variable, the mass rate is $dm/dt = S(T_{\max} - T_{\min})\sigma$ so that the fuel used per range, the performance criterion, is

$$J = \frac{1}{r_f}\int_o^{r_f}\left[\frac{S(T_{\max} - T_{\min})\sigma}{Ma\cos\gamma F_f}\right]\left(1 + \frac{h}{R_0}\right)dr \tag{40}$$

For additional realism, the thrust-force direction is fixed on the vehicle. Therefore, T is replaced by $T\cos(\alpha + \alpha_T)$ in Eq. (2) and

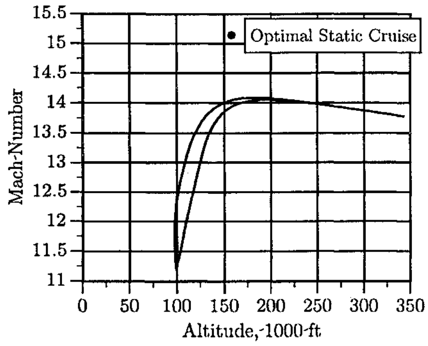

Fig. 6 Steady-state point and optimal periodic trajectory (NASA Dryden model).

$T \sin(\alpha + \alpha_T)$ is introduced into Eq. (3), where α_T is the thrust-offset angle. Note that the Legendre–Clebsch condition cannot be satisfied for values of S interior to its control set. There can be no singular arc including static cruise. However, if α becomes a state variable dynamically related to a new control variable through an actuator, then for extremal controls lying in their interiors of their control sets, the Legendre–Clebsch condition is satisfied in weak form, and again there is a possibility of a doubly singular arc.

To avoid excessive aerodynamic loading, an upper bound on the total loading experienced during the orbit is imposed as the inequality constraint

$$F(h, M, \gamma, C_L, S) = \left\{[T \cos(\alpha + \alpha_T) - D]^2 + [T \sin(\alpha + \alpha_T) + L]^2\right\}^{\frac{1}{2}} / W - g_{\max} \leq 0$$

where the number of g is defined as the magnitude of the net applied force divided by the vehicle weight and $g_{\max}$ is the maximum vehicle acceleration in g. The fuel optimal periodic control problem is to find the control variable $C_L(\cdot)$ and $S(\cdot)$, the initial conditions $h(0)$, $M(0)$, and $\gamma(0)$, and the period r_f that minimizes Eq. (40) subject to the differential equations (1–3), the inequality constraints on F and S and the periodicity constraints (8).

No limits on the number of switches were imposed. However, even though the number of switches over the orbit is found to be only two, S does take on values interior to its constraints set when the g-constraint becomes active.

The numerical solution to this hypersonic cruise optimization problem produced a periodic solution yielding a 10.8% improvement over the static cruise path where the maximum g loading is 5. Figure 6 shows M vs h over the orbit, as well as the (h, M) pair corresponding to steady-state cruise. The large range of altitude of the periodic trajectory is noteworthy, obtaining a maximum altitude of about 340,000 ft and a minimum altitude of about 90,000 ft. Because of the large atmospheric density changes, the orbit is composed of two flight regimes: a Keplerian arc at high altitudes, because aerodynamic forces are largely absent, and dominant aerodynamic forces at low altitudes. The aerodynamic maneuver consists of an unpowered glide to minimum altitude, then a powered climb out of the atmosphere. Such a cruise trajectory is believed superior to the steady-state path, partially because of its ability to radiate heat at high altitudes, and the total absorbed heat is about half that for static cruise.[32]

VI. Regulators for Optimal Periodic Processes

To mechanize these optimal periodic processes in the presence of state derivations from the optimal periodic path and system parameter uncertainties, a feedback controller called the periodic regulator has been developed.[26,32,33] From a theoretical viewpoint the special characteristics of the periodic optimal control path must be taken into account. The regulator problem is formulated by considering the accessory minimum problem in the calculus of variations about the infinitely repeated periodic orbit. The accessory minimum problem is determined by expanding the cost criterion to second order and the dynamics to first order in terms of the state and control variations from the nominal optimal periodic path. In the resulting linear quadratic control problem, the second-variational cost criterion is to be minimized subject to the linearized equation of motion. It is shown that the state variation based on the resulting periodic regulator converges in an $(n-1)$-dimensional locally stable subspace in the neighborhood of the periodic orbit. The remaining one-dimensional subspace is avoided by determining an index point on the normal periodic path for which the state variation is defined, such that the projection of the state variation onto the orbital velocity vector is zero. Recall that the velocity vector along the periodic path is an eigenvector with an associated neutrally stable eigenvalue.

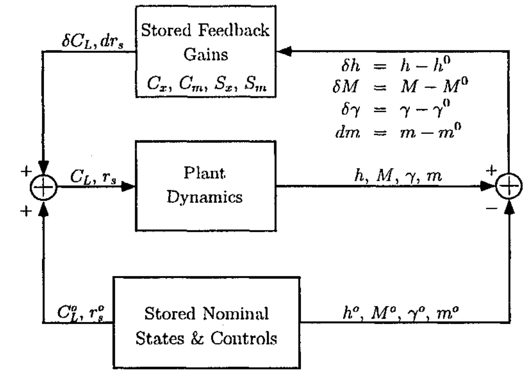

Fig. 7 Control diagram of periodic regulator.

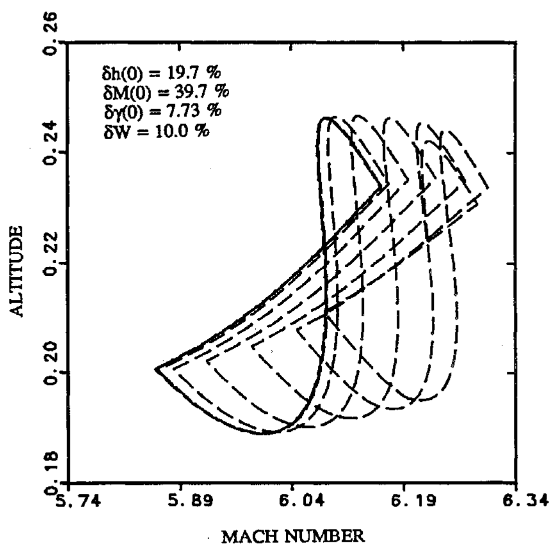

Fig. 8 Periodic hypersonic cruise regulator performance in altitude (5×10^5) vs Mach number (NASA Langley model).

A central assumption in the determination of periodic optimal paths is that the original system is autonomous. This certainly is not the case in the cruise problem where mass changes in the dynamic system occur because of fuel usage. However, if the parameter such as mass can be considered adiabatic, i.e., the parameters are approximately constant over one given period but vary somewhat over multiple periods, then this paramater change can be included in the periodic regulator. The essential feature is to ensure that with this change in mass, all first-order necessary conditions are approximately met in the neighboring optimum path.

A periodic regulator for mechanizing periodic cruise is shown in Fig. 7. Note that although variations in C_L attributable to the variations in the state and parameter are computed, the thrust switch ranges, rather than the thrust, are used as control parameters, and the change in the switch range is determined as a linear function of the state and parameter variations.[26,32,33] The controller is of the form

$$\delta C_L = C_x \delta x + C_m \, \mathrm{d}m$$

$$\mathrm{d}r_s = S_x \delta x + S_m \, \mathrm{d}m$$

where $C_x(r_I)$, $C_m(r_I)$, $S_x(r_I)$, and $S_m(r_I)$ are the controller gains and $\delta x(r) \equiv x(r) - x^{\circ}(r_I)$, where r_I is the index range on the

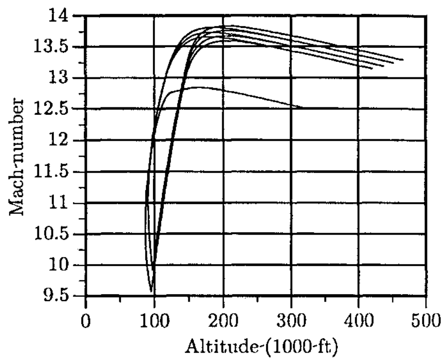

Fig. 9 Periodic hypersonic cruise regulator performance in altitude vs Mach number (NASA Dryden model).

nominal path such that $\delta x(r)$ belongs to the stable subspace. Based on the nominal path constructed from data obtained from NASA Langley Research Center[17] and given in Fig. 5, asymptotically stable paths are shown for particular state and mass changes in Fig. 8. The performance of the optimal periodic regulator for the more energetic periodic optimal normal path (Fig. 6) based on the NASA Dryden model is shown in Fig. 9.[32] The optimal regulator includes a 6-g constraint and mass changes over each orbit of 3%.

VII. Conclusions

This paper integrates into a coherent presentation a variety of work performed over the past 20 years. Theoretical results on necessity and sufficiency for weak local optimality of periodic orbits have gone hand in hand with application of the theory to periodic optimal fight. As the hypersonic-vehicle results show, optimal periodic orbits can produce significant fuel savings. These savings are obtained for hypersonic aircraft designs that were possibly optimized for static cruise. Designs that are enhanced by periodic flight may produce very significant results. For example, are hypersonic vehicles configured as wave riders best for fuel cruise or would an accelerator be best where large portions of the flight path remain outside the atmosphere (see Fig. 6). Other aspects not considered are heat rates and total heating that are incurred in these very different flight paths. The emphasis here has been on fuel per range cruise. However, periodic flight paths appear to produce significant improvement when applied to endurance. An objection to these paths has been the apparent discomfort for manned flight. Although it might be argued that the flight paths might be made more gentle and thereby suboptimal, the use of autonomous unmanned aircraft, which is becoming ever more popular, is a possible application of these ideas. This possibility is further enhanced by the ability to mechanize these flight paths using the periodic regulator through the enormous capability of the modern microprocessor.

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